Cross-over Phenomena

Consider the following practical problem: one is trying to extract the critical exponents of an Ising system.

For extracting β, one needs to set $h=0$, vary t and measure magnetization $m \sim (-t)^{\beta}$. Similarly, to extract δ, one instead needs to set $t=0$, vary h and measure magnetization $m \sim h^{1/\delta}$. However, in a realistic setup one will always have stray fields so that $h \neq 0$ and further, one will not know the exact T_c so that $t \neq 0$. One might naively expect that as long as $h, t \ll 1$, one will be able to extract the critical exponents correctly. This is incorrect – as we now show that depending on whether $h/t^{\beta\delta} \gg 1$ or $\frac{h}{t^{\beta\delta}} \ll 1$, one will or will not obtain the critical exponent one is after.

Again, lets start from the scaling of free energy $f_s \sim b^{-d} f_s(t\, b^{y_t}, h\, b^{y_h})$

$$\Rightarrow \quad f_s \sim t^{d/y_t}\, \phi(h\, t^{-y_h/y_t})$$

$$\Rightarrow \quad m = \frac{\partial f_s}{\partial h} \sim t^{\frac{d-y_h}{y_t}}\, \phi'(h\, t^{-y_h/y_t})$$

To see the behavior $m \sim (-t)^{\beta}$, one needs that $h\, t^{-y_h/y_t} \ll 1$, so that

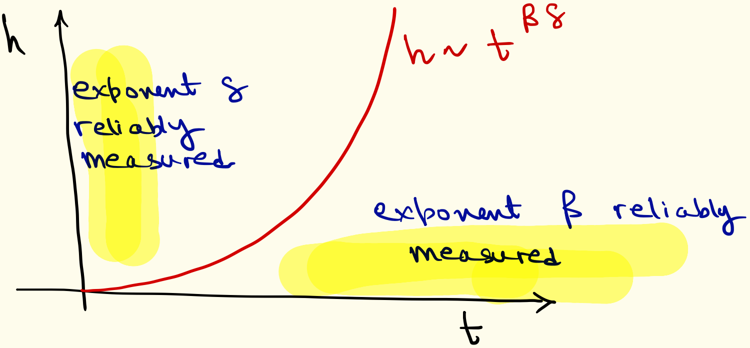
$$m \sim t^{(d-y_h)/y_t}\, \phi'(0) \sim t^{(d-y_h)/y_t}$$

In the opposite limit, $h\, t^{-y_h/y_t} \gg 1$, m is governed by $\phi'(\infty)$. Since we know that at $t=0$, m must depend in a power law fashion on h, $\Rightarrow \phi'(x \gg 1) \sim x^{(d-y_h)/y_h}$

so that the dependence on t is cancelled out. In this limit, therefore $m \sim h^{(d-y_h)/y_h}$

$= h^{1/\delta}$ as expected.

Therefore, when $\frac{h}{t^{y_h/y_t}} = \frac{h}{t^{\beta\delta}} \ll 1$, one reliably obtains exponent β while in the opposite limit, $\frac{h}{t^{\beta\delta}} \gg 1$, one reliably obtains the exponent δ.



What happens if one measures the specific heat in the regime $h \gg t^{\beta\delta}$ as a function of temperature at fixed h? As just discussed, in this regime, $f_s \sim t^{d/y_t} \phi(h t^{-y_h/y_t})$ with

$$\phi(x) \sim x^{\frac{d-y_h}{y_h}+1} \sim x^{d/y_h}.$$

$$\Rightarrow \quad f_s \sim t^{\frac{d}{y_t} - \frac{d y_h}{y_h y_t}} \sim t^0$$ at the leading order $\Rightarrow$ specific heat does not diverge and one erroneously concludes that exponent $\alpha = 0$.

Returning to the φ^4 theory in $d < 4$ dimensions, close the Gaussian fixed point, the free energy may be written as

$$f_s \sim b^{-d} f_s(t\, b^{y_t}, u\, b^{y_u})$$

where $y_t = 2$ and $y_u = 4-d$. Setting $t\, b^{y_t} \sim 1$,

$$\Rightarrow \quad f_s \sim t^{d/y_t} f_s(u\, t^{-y_u/y_t}).$$

Based on our discussion above, one will see the Gaussian exponents if $u|t|^{-y_u/y_t} \ll 1$, i.e., if $\boxed{|t|^{2-d/2} \gg u}$, which is precisely the Ginzburg criteria we derived earlier based on the magnitude of fluctuations around the mean-field.